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1997 J. Phys. A: Math. Gen. 30 5877

(http://iopscience.iop.org/0305-4470/30/16/027)

J. Phys. A: Math. Gen. **30** (1997) 5877–5885. Printed in the UK PII: S0305-4470(97)81548-X

Some orthogonal very well poised ${}_8\varphi_7$-functions

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Received 4 February 1997

Abstract. Recently Ismail, Masson and Suslov established a continuous orthogonality relation and some other properties of a ${}_2\varphi_1$-Bessel function on a q-quadratic grid. Askey suggested that the 'Bessel-type orthogonality' found in the above paper at the ${}_2\varphi_1$-level really has a general character and can be extended up to the ${}_8\varphi_7$-level. Very well poised ${}_8\varphi_7$-fuctions are known as a nonterminating version of the classical Askey–Wilson polynomials. In this paper we prove Askey's conjecture and discuss some properties of the orthogonal ${}_8\varphi_7$-functions. Another type of the orthogonality relation for a very well poised ${}_8\varphi_7$-function was recently found by Askey, Rahman and Suslov.

1. Introduction

The Askey–Wilson polynomials [4] are

$$\begin{aligned} p_n(x) &= p_n(x; a, b, c, d) \\ &= a^{-n}(ab, ac, ad; q)_n\, {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{\mathrm{i}\theta}, ae^{-\mathrm{i}\theta} \\ ab, ac, ad \end{matrix}; q, q\right) \end{aligned} \tag{1.1}$$

where $x = \cos\theta$. These polynomials are known as the most general system of classical orthogonal polynomials (see[1, 4–7, 15, 16]).

The symbol ${}_4\varphi_3$ in (1.1) is a special case of basic hypergeometric series [10],

$$\begin{aligned} {}_r\varphi_s(t) &:= {}_r\varphi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, t\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} ((-1)^n q^{n(n-1)/2})^{1+s-r} t^n. \end{aligned} \tag{1.2}$$

The standard notations for the q-shifted factorials are

$$(a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}) \tag{1.3}$$

$$(a_1, a_2, \ldots, a_r; q)_n := \prod_{k=1}^{r} (a_k; q)_n \tag{1.4}$$

and

$$(a; q)_\infty := \lim_{n\to\infty} (a; q)_n \tag{1.5}$$

$$(a_1, a_2, \ldots, a_r; q)_\infty := \prod_{k=1}^{r} (a_k; q)_\infty \tag{1.6}$$

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provided $|q| < 1$. For an excellent account on the theory of basic hypergeometric series see [10].

Askey and Wilson found the orthogonality relation

$$\int_0^\pi \frac{p_n(\cos\theta; a, b, c, d)\, p_m(\cos\theta; a, b, c, d)(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}, b\mathrm{e}^{\mathrm{i}\theta}, b\mathrm{e}^{-\mathrm{i}\theta}, c\mathrm{e}^{\mathrm{i}\theta}, c\mathrm{e}^{-\mathrm{i}\theta}, d\mathrm{e}^{\mathrm{i}\theta}, d\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty}\, \mathrm{d}\theta$$
$$= \delta_{nm} \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty} \times \frac{(1 - abcdq^{-1})(q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - abcdq^{2n-1})(abcdq^{-1}; q)_n}. \tag{1.7}$$

In the fundamental paper [4], they studied in details many other properties of these polynomials.

Recently Ismail *et al* [12, 13] have considered the ${}_2\varphi_1$-function,

$$J_\nu(z, r) = \widetilde{J}_\nu(x(z), r|q)$$
$$:= \left(\frac{r}{2}\right)^\nu \frac{(q^{\nu+1}, -r^2/4; q)_\infty}{(q; q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} q^{(\nu+1)/2}\mathrm{e}^{\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{-\mathrm{i}\theta} \\ q^{\nu+1} \end{matrix}; q, -\frac{r^2}{4}\right) \tag{1.8}$$

as a q-analogue on a q-quadratic grid of the Bessel function [22],

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-x^2/4)^n}{n!\,\Gamma(\nu + n + 1)}. \tag{1.9}$$

They established the following orthogonality property for the q-Bessel function,

$$\int_0^\pi \widetilde{J}_\nu(\cos\theta, r)\widetilde{J}_\nu(\cos\theta, r') \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(q^\alpha\mathrm{e}^{\mathrm{i}\theta}, q^\alpha\mathrm{e}^{-\mathrm{i}\theta}, q^{1-\alpha}\mathrm{e}^{\mathrm{i}\theta}, q^{1-\alpha}\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty}$$
$$\times (q^{(\nu+1)/2}\mathrm{e}^{\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{-\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1}\, \mathrm{d}\theta = 0 \tag{1.10}$$

if $r \neq r'$,

$$\int_0^\pi (\widetilde{J}_\nu(\cos\theta, r))^2 \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(q^\alpha\mathrm{e}^{\mathrm{i}\theta}, q^\alpha\mathrm{e}^{-\mathrm{i}\theta}, q^{1-\alpha}\mathrm{e}^{\mathrm{i}\theta}, q^{1-\alpha}\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty}$$
$$\times (q^{(\nu+1)/2}\mathrm{e}^{\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{-\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{\mathrm{i}\theta}, q^{(\nu+1)/2}\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1}\, \mathrm{d}\theta$$
$$= \frac{-4\pi(1 - q)q^{-(\nu+1)/2}}{(q, q^{(\nu+1)/2+\alpha}, q^{(\nu+1)/2-\alpha+1}; q)_\infty^2} \frac{\partial \widetilde{J}_\nu(x(\alpha), r)}{\partial r^2} \frac{\nabla \widetilde{J}_\nu(x(\alpha), r)}{\nabla x(\alpha)} \tag{1.11}$$

if $r = r'$. Here r and r' are two roots of the equation

$$J_\nu(\alpha, r) = J_\nu(\alpha, r') = 0 \tag{1.12}$$

and $J_\nu(z, r) = \widetilde{J}_\nu(x(z), r)$, $x(z) = \frac{1}{2}(q^z + q^{-z})$, ($x = \cos\theta$, if $q^z = \mathrm{e}^{\mathrm{i}\theta}$), $\mathrm{Re}\,\nu > -1$, and $0 < \mathrm{Re}\,\alpha < 1$. (See [12, 13] for more details.) This is a q-version of the orthogonality relation for the classical Bessel function

$$\int_0^1 x J_\nu(rx) J_\nu(r'x)\, \mathrm{d}x = \begin{cases} 0 & \text{if } r \neq r' \\ \frac{1}{2}(J_{\nu+1}(r))^2 & \text{if } r = r' \end{cases} \tag{1.13}$$

under the conditions $J_\nu(r) = J_\nu(r') = 0$ [22].

Askey [2] suggested that the orthogonality relation (1.10), (1.11) can be extended to the level of very well poised ${}_8\varphi_7$-functions. Our main objective in this paper is to prove his conjecture.

It is also worth mentioning that recently Bustoz and Suslov [9] established a similar orthogonality property for basic trigonometric functions and introduced the corresponding q-Fourier series.

2. Difference equation and its ${}_8\varphi_7$-Solutions

Let us consider a *difference equation of hypergeometric type*

$$\sigma(z)\frac{\Delta}{\nabla x_1(z)}\left(\frac{\nabla u(z)}{\nabla x(z)}\right)+\tau(z)\frac{\Delta u(z)}{\Delta x(z)}+\lambda u(z)=0 \tag{2.1}$$

on a q-quadratic lattice $x(z)=\frac{1}{2}(q^z+q^{-z})$ with $x_1(z)=x(z+\frac{1}{2})$ and $\Delta f(z)=\nabla f(z+1)=f(z+1)-f(z)$. Here, in the most general case,

$$\sigma(z)=q^{-2z}(q^z-a)(q^z-b)(q^z-c)(q^z-d) \tag{2.2}$$

$$\begin{aligned}\tau(z)&=\frac{\sigma(-z)-\sigma(z)}{\nabla x_1(z)}\\ &=\frac{2q^{1/2}}{1-q}(abc+abd+acd+bcd-a-b-c-d+2(1-abcd)x)\end{aligned} \tag{2.3}$$

$$\lambda=\lambda_\nu=\frac{4q^{3/2}}{(1-q)^2}(1-q^{-\nu})(1-abcdq^{\nu-1}). \tag{2.4}$$

Equation (2.1) can also be rewritten in self-adjoint form,

$$\frac{\Delta}{\nabla x_1(z)}\left(\sigma(z)\rho(z)\frac{\nabla u(z)}{\nabla x(z)}\right)+\lambda\rho(z)u(z)=0 \tag{2.5}$$

where $\rho(z)$ is a solution of the *Pearson equation*,

$$\Delta(\sigma(z)\rho(z))=\tau(z)\rho(z)\nabla x_1(z). \tag{2.6}$$

See [16, 20] for details.

As is well known, there are different kinds of solutions of equation (2.1). For integer values of the parameter $\nu=n=0,1,2,\ldots,$ the famous solutions of (2.1) are the Askey–Wilson ${}_4\varphi_3$-polynomials (1.1) ([4, 10, 7]). For arbitrary values of this parameter, solutions of (2.1) can be written in terms of ${}_8\varphi_7$-functions [6, 14, 20]. Let us choose the following solution,

$$\begin{aligned}u_\nu(z)=u_\nu(z;a,b,c,d)&=\frac{(qa/b,q^{1-\nu+z}/b,q^{1-\nu-z}/b;q)_\infty}{(q^{1-\nu}a/b,q^{1+z}/b,q^{1-z}/b;q)_\infty}\\ &\quad\times{}_8\varphi_7\left(\begin{matrix}\frac{aq^{-\nu}}{b},q\sqrt{\frac{aq^{-\nu}}{b}},-q\sqrt{\frac{aq^{-\nu}}{b}},q^{-\nu},\frac{q^{1-\nu}}{bc},\frac{q^{1-\nu}}{bd},aq^z,aq^{-z}\\ \sqrt{\frac{aq^{-\nu}}{b}},-\sqrt{\frac{aq^{-\nu}}{b}},\frac{aq}{b},ac,ad,\frac{q^{1-\nu+z}}{b},\frac{q^{1-\nu-z}}{b}\end{matrix};q,cdq^\nu\right)\end{aligned} \tag{2.7}$$

$$\begin{aligned}&=\frac{(q^{1-\nu}/ab;q)_\infty}{(q/ab;q)_\infty}\,{}_4\varphi_3\left(\begin{matrix}q^{-\nu},abcdq^{\nu-1},aq^z,aq^{-z}\\ ab,ac,ad\end{matrix};q,q\right)\\ &\quad+\frac{(q^{-\nu},abcdq^{\nu-1},qc/b,qd/b;q)_\infty}{(ac,ad,cdq^\nu,ab/q;q)_\infty}\frac{(aq^z,aq^{-z};q)_\infty}{(q^{1+z}/b,q^{1-z}/b;q)_\infty}\\ &\quad\times{}_4\varphi_3\left(\begin{matrix}q^{1-\nu}/ab,cdq^\nu,q^{1+z}/b,q^{1-z}/b\\ qc/b,qd/b,q^2/ab\end{matrix};q,q\right).\end{aligned} \tag{2.8}$$

We have used (III.36) of [10] to transform (2.7) into (2.8). Both ${}_4\varphi_3$-functions in (2.8) are balanced and converge when $|q|<1$, the ${}_8\varphi_7$-function in (2.7) has a very well poised structure [10]. A similar ${}_8\varphi_7$-function was recently discussed by Rahman [17].

One can easily see, that for integers $\nu=n=0,1,2,\ldots$ the ${}_8\varphi_7$-solution above is just a multiple of the Askey–Wilson polynomial (1.1). Otherwise, the essential poles of this ${}_8\varphi_7$-solution coincide with the simple poles of $(q^{1+z}/b, q^{1-z}/b; q)_\infty^{-1}$. Therefore, the function

$$v_\nu(z) = v_\nu(z; a, b, c, d) := (q^{1+z}/b, q^{1-z}/b; q)_\infty u_\nu(z; a, b, c, d) \tag{2.9}$$

where $u_\nu(z; a, b, c, d)$ is defined by (2.7), (2.8), is an entire function in the complex z-plane.

3. Solution of Pearson equation

In order to rewrite equation (2.1) for the function (2.7), (2.8) in the self-adjoint form (2.5), we have to find a *solution of the Pearson-type equation* (2.6). In the case of the q-quaratic grid $x = \frac{1}{2}(q^z + q^{-z})$ this equation can be rewritten in the form

$$\begin{aligned}\frac{\rho(z+1)}{\rho(z)} &= \frac{\sigma(-z)}{\sigma(z+1)} \\ &= q^{-4z-2}q^{2z+1}\frac{(1-aq^z)(1-q^{-z}/b)(1-cq^z)(1-dq^z)}{(1-aq^{-z-1})(1-q^{z+1}/b)(1-cq^{-z-1})(1-dq^{-z-1})}.\end{aligned} \tag{3.1}$$

It is easy to check that

$$\frac{\rho_0(z+1)}{\rho_0(z)} = q^{-4z-2} \qquad \text{for } \rho_0(z) = \frac{(q^{2z}, q^{-2z}; q)_\infty}{q^z - q^{-z}} \tag{3.2}$$

$$\frac{\rho_\alpha(z+1)}{\rho_\alpha(z)} = q^{-2z-1} \qquad \text{for } \rho_\alpha(z) = (\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha; q)_\infty \tag{3.3}$$

$$\frac{\rho_a(z+1)}{\rho_a(z)} = \frac{1-aq^{-z-1}}{1-aq^z} \qquad \text{for } \rho_a(z) = (aq^z, aq^{-z}; q)_\infty. \tag{3.4}$$

(See, for example, [16, 18, 19] for methods of solving the Pearson equation.) Therefore, one can choose the following solution of (3.1),

$$\rho(z) = \frac{(q^z - q^{-z})^{-1}(q^{2z}, q^{-2z}, q^{1+z}/b, q^{1-z}/b; q)_\infty}{(\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha, aq^z, aq^{-z}, cq^z, cq^{-z}, dq^z, dq^{-z}; q)_\infty} \tag{3.5}$$

where α is an arbitrary additional parameter. In the next section we shall see that this solution satisfies the correct boundary conditions for the second-order divided-difference Askey–Wilson operator (2.5) for certain values of this parameter α.

Special cases $\alpha = b$ or $\alpha = q/b$ of (3.5) give us the weight function for the Askey–Wilson polynomials (cf [4, 7]).

4. Orthogonality Property

We can now establish the orthogonality relation of the ${}_8\varphi_7$-functions (2.7) with respect to the weight function (3.5). Let us apply the following q-version of the Sturm–Liouville procedure (cf [7, 8, 16]). Consider the difference equations for the functions $u_\nu(z) = u_\nu(z; a, b, c, d)$ and $u_\mu(z) = u_\mu(z; a, b, c, d)$ in self-adjoint form,

$$\frac{\Delta}{\nabla x_1(z)}\left(\sigma(z)\rho(z)\frac{\nabla u_\mu(z)}{\nabla x(z)}\right) + \lambda_\mu \rho(z) u_\mu(z) = 0 \tag{4.1}$$

$$\frac{\Delta}{\nabla x_1(z)}\left(\sigma(z)\rho(z)\frac{\nabla u_\nu(z)}{\nabla x(z)}\right) + \lambda_\nu \rho(z) u_\nu(z) = 0 \tag{4.2}$$

where eigenvalues $\lambda = \lambda_\nu$ and $\lambda' = \lambda_\mu$ are defined by (2.4). Let us multiply the first equation by $u_\nu(z)$, the second one by $u_\mu(z)$, and subtract the second equality from the first one. As a result we obtain

$$(\lambda_\mu - \lambda_\nu)u_\mu(z)u_\nu(z)\rho(z)\nabla x_1(z) = \Delta[\sigma(z)\rho(z)W(u_\mu(z), u_\nu(z))] \tag{4.3}$$

where

$$\begin{aligned} W(u_\mu(z), u_\nu(z)) &= \begin{vmatrix} u_\mu(z) & u_\nu(z) \\ \frac{\nabla u_\mu(z)}{\nabla x(z)} & \frac{\nabla u_\nu(z)}{\nabla x(z)} \end{vmatrix} \\ &= u_\mu(z)\frac{\nabla u_\nu(z)}{\nabla x(z)} - u_\nu(z)\frac{\nabla u_\mu(z)}{\nabla x(z)} \\ &= \frac{u_\nu(z)u_\mu(z-1) - u_\mu(z)u_\nu(z-1)}{x(z) - x(z-1)} \end{aligned} \tag{4.4}$$

is the analogue of the Wronskian [16].

We need to know the pole structure of the analogue of the Wronskian $W(u_\mu, u_\nu)$ in (4.4). Let us transform the u's to the entire functions v's by (2.9),

$$u_\varepsilon(z) = \varphi(z)v_\varepsilon(z) \tag{4.5}$$

where $\varepsilon = \mu, \nu$ and

$$\varphi(z) = (q^{1+z}/b, q^{1-z}/b; q)_\infty^{-1}. \tag{4.6}$$

Thus,

$$W(u_\mu(z), u_\nu(z)) = \varphi(z)\varphi(z-1)W(v_\mu(z), v_\nu(z)) \tag{4.7}$$

where the new 'Wronskian', $W(v_\mu(z), v_\nu(z))$, is clearly an entire function in z.

Integrating (4.3) over the contour C indicated in figure 1; where the variable z is such that $z = \mathrm{i}\theta/\log^{-1} q$ and $-\pi \leqslant \theta \leqslant \pi$; gives

$$(\lambda_\nu - \lambda_\mu)\int_C u_\nu(z)u_\mu(z)\rho(z)\nabla x_1(z)\,\mathrm{d}z = \int_C \Delta[\sigma(z)\rho(z)\varphi(z)\varphi(z-1)W(v_\nu(z), v_\mu(z))]\,\mathrm{d}z. \tag{4.8}$$

All poles of the integrand on the right-hand side of (4.8) coincide with the simple poles of the function

$$\begin{aligned} \sigma(z)\rho(z)\varphi(z)\varphi(z-1) = &-\frac{b(q^{2z}, q^{1-2z}; q)_\infty}{(\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha)_\infty} \\ &\times(aq^z, aq^{1-z}, q^{1+z}/b, q^{2-z}/b, cq^z, cq^{1-z}, dq^z, dq^{1-z}; q)_\infty^{-1}. \end{aligned} \tag{4.9}$$

The integrand on the right-hand side of (4.8) has the natural purely imaginary period $T = 2\pi\mathrm{i}/\log^{-1} q$ when $0 < q < 1$, so this integral is equal to

$$\int_D [\sigma(z)\varphi(z)\varphi(z-1)\rho(z)W(v_\nu(z), v_\mu(z))]\,\mathrm{d}z \tag{4.10}$$

where D is the boundary of the rectangle in the figure oriented anticlockwise. The analogue of the Wronskian here is an entire function. Thus, the essential poles of the integrand in (4.10) coincide with the poles of function (4.9) and are just the simple poles at $z = \alpha_0$ and $z = 1 - \alpha_0$, where $q^{\alpha_0} = \alpha$. Evaluation of the residues at these simple poles gives

$$\begin{aligned} \int_C u_\nu(z)u_\mu(z)\rho(z)\nabla x_1(z)\,\mathrm{d}z = &\frac{b(q/\alpha b, \alpha/b; q)_\infty}{(q, q, \alpha a, qa/\alpha, \alpha c, qc/\alpha, \alpha d, qd/\alpha; q)_\infty} \\ &\times\frac{-4\pi\mathrm{i}}{\log q}\frac{W(u_\nu(\alpha_0), u_\mu(\alpha_0))}{\lambda_\nu - \lambda_\mu} \end{aligned} \tag{4.11}$$

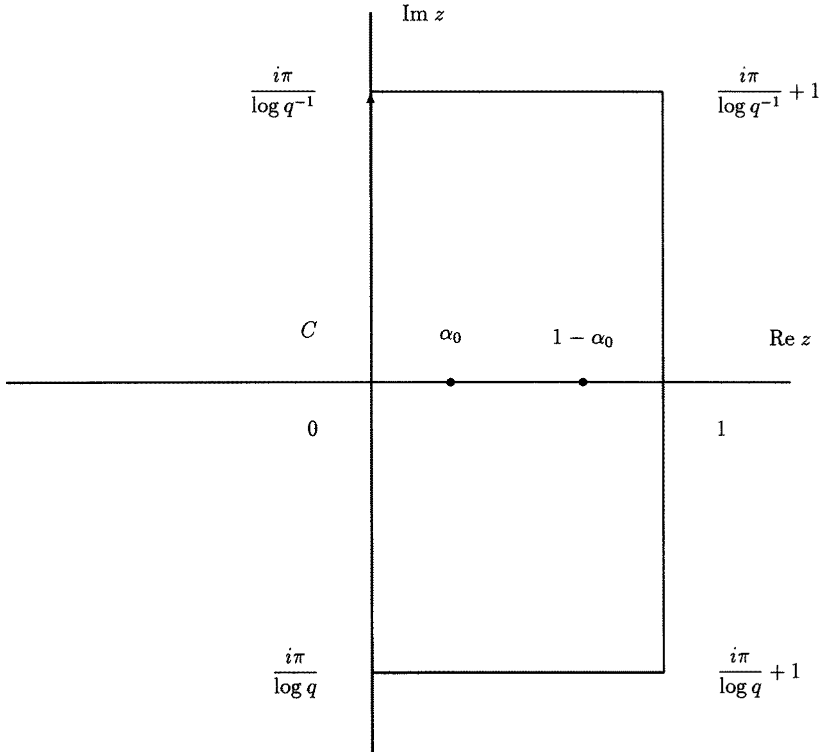

Figure 1.

if $\nu \neq \mu$ and $0 < \mathrm{Re}\,\alpha_0 < \frac{1}{2}$.

When $\alpha = b$ or $\alpha = q/b$ and the parameters $\nu, \mu = 0, 1, 2, \ldots$ are nonnegative integers, equations (2.8) and (4.11) imply the orthogonality relation for the Askey–Wilson polynomials (1.7).

If both parameters μ and ν are not nonnegative integers, we can rewrite (4.11) as

$$\begin{aligned}\int_0^\pi \widetilde{v}_\nu(\cos\theta)\widetilde{v}_\mu(\cos\theta)\frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(\alpha\mathrm{e}^{\mathrm{i}\theta}, \alpha\mathrm{e}^{-\mathrm{i}\theta}, q\mathrm{e}^{\mathrm{i}\theta}/\alpha, q\mathrm{e}^{-\mathrm{i}\theta}/\alpha)_\infty}\\ \times(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}, q\mathrm{e}^{\mathrm{i}\theta}/b, q\mathrm{e}^{-\mathrm{i}\theta}/b, c\mathrm{e}^{\mathrm{i}\theta}, c\mathrm{e}^{-\mathrm{i}\theta}, d\mathrm{e}^{\mathrm{i}\theta}, d\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1}\,\mathrm{d}\theta\\ = (q, q, \alpha a, qa/\alpha, q\alpha/b, q^2/\alpha b, \alpha c, qc/\alpha, \alpha d, qd/\alpha; q)_\infty^{-1}\\ \times\frac{-4\pi q^{1/2}b}{1-q}\,\frac{W(\widetilde{v}_\nu(\frac{1}{2}(\alpha+\alpha^{-1})), \widetilde{v}_\mu(\frac{1}{2}(\alpha+\alpha^{-1})))}{\lambda_\nu-\lambda_\mu}.\end{aligned} \tag{4.12}$$

Here we use the notation $\widetilde{v}_\varepsilon(x(z)) = v_\varepsilon(z)$, $\varepsilon = \mu, \nu$ and $x(z) = \frac{1}{2}(q^z + q^{-z}) = \cos\theta$ if $q^z = \mathrm{e}^{\mathrm{i}\theta}$; we also assume that $\max(|a|, |q/b|, |c|, |d|) < 1$.

Choosing the parameters μ and ν as ε-solutions of the equation

$$\widetilde{v}_\varepsilon(\tfrac{1}{2}(\alpha+\alpha^{-1}); a, b, c, d) = 0 \tag{4.13}$$

we finally arrive at the *orthogonality relation* of the ${}_8\varphi_7$-functions,

$$\begin{aligned}\int_0^\pi \widetilde{v}_\nu(\cos\theta)\widetilde{v}_\mu(\cos\theta)\frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(\alpha\mathrm{e}^{\mathrm{i}\theta}, \alpha\mathrm{e}^{-\mathrm{i}\theta}, q\mathrm{e}^{\mathrm{i}\theta}/\alpha, q\mathrm{e}^{-\mathrm{i}\theta}/\alpha)_\infty}\\ \times(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}, q\mathrm{e}^{\mathrm{i}\theta}/b, q\mathrm{e}^{-\mathrm{i}\theta}/b, c\mathrm{e}^{\mathrm{i}\theta}, c\mathrm{e}^{-\mathrm{i}\theta}, d\mathrm{e}^{\mathrm{i}\theta}, d\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1}\,\mathrm{d}\theta = 0\end{aligned} \tag{4.14}$$

if $\mu \neq \nu$, and

$$\int_0^{\pi} (\widetilde{v}_\nu(\cos\theta))^2 \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(\alpha\mathrm{e}^{\mathrm{i}\theta}, \alpha\mathrm{e}^{-\mathrm{i}\theta}, q\mathrm{e}^{\mathrm{i}\theta}/\alpha, q\mathrm{e}^{-\mathrm{i}\theta}/\alpha)_\infty}$$
$$\times(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}, q\mathrm{e}^{\mathrm{i}\theta}/b, q\mathrm{e}^{-\mathrm{i}\theta}/b, c\mathrm{e}^{\mathrm{i}\theta}, c\mathrm{e}^{-\mathrm{i}\theta}, d\mathrm{e}^{\mathrm{i}\theta}, d\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty^{-1} d\theta$$
$$= (q, q, \alpha a, qa/\alpha, q\alpha/b, q^2/\alpha b, \alpha c, qc/\alpha, \alpha d, qd/\alpha; q)_\infty^{-1}$$
$$\times \frac{-4\pi q^{1/2} b}{1-q} \left(\frac{\partial \widetilde{v}_\nu}{\partial \lambda_\nu} \frac{\nabla \widetilde{v}_\nu}{\nabla x} \right)\bigg|_{q^z=\alpha} \tag{4.15}$$

if $\mu = \nu$. We shall assume throughout this paper that $q^{1/2} < \alpha < 1$, $0 < a < 1$, $0 < q/b < 1$, $0 < c < 1$, and $0 < d < 1$.

5. Some properties of zeros

In section 4 we established the orthogonality relation of the ${}_8\varphi_7$-functions (2.7)–(2.9) under the boundary condition (4.13). Here we would like to discuss some properties of ν-zeros of the function

$$\widetilde{v}_\nu(\tfrac{1}{2}(\alpha+\alpha^{-1}); a, b, c, d) = \frac{(qa/b, \alpha q^{1-\nu}/b, q^{1-\nu}/\alpha b; q)_\infty}{(q^{1-\nu}a/b; q)_\infty}$$
$$\times\ {}_8\varphi_7\left(\begin{matrix} \frac{aq^{-\nu}}{b}, q\sqrt{\frac{aq^{-\nu}}{b}}, -q\sqrt{\frac{aq^{-\nu}}{b}}, q^{-\nu}, \frac{q^{1-\nu}}{bc}, \frac{q^{1-\nu}}{bd}, \alpha a, a/\alpha \\ \sqrt{\frac{aq^{-\nu}}{b}}, -\sqrt{\frac{aq^{-\nu}}{b}}, \frac{aq}{b}, ac, ad, \frac{q^{1-\nu}\alpha}{b}, \frac{q^{1-\nu}}{\alpha b} \end{matrix} ; q, cdq^\nu \right) \tag{5.1}$$
$$= \frac{(cd, q\alpha/b, q^\nu acd/\alpha, q^{1-\nu}/\alpha b; q)_\infty}{(acd/\alpha, cdq^\nu; q)_\infty}$$
$$\times\ {}_8\varphi_7\left(\begin{matrix} \frac{acd}{q\alpha}, \sqrt{\frac{acd}{q^{1/2}\alpha}}, -\sqrt{\frac{acd}{q^{1/2}\alpha}}, q^{-\nu}, abcdq^{\nu-1}, \frac{a}{\alpha}, \frac{c}{\alpha}, \frac{d}{\alpha} \\ \sqrt{\frac{acd}{q\alpha}}, -\sqrt{\frac{acd}{q\alpha}}, \frac{q^\nu acd}{\alpha}, \frac{q^{1-\nu}}{\alpha b}, ac, ad, cd \end{matrix} ; q, q\frac{\alpha}{b} \right). \tag{5.2}$$

We have used (III.23) of [10] to transform (5.1) into (5.2).

The main properties of zeros of the function (5.1), (5.2) can be investigated by using the same methods as in [9, 12, 13]. The first property is that the function $\widetilde{v}_\nu(x)$ in (5.1), (5.2) has an infinity of real ν-zeros. In order to see that, one can consider the large ν-asymptotics of the ${}_8\varphi_7$-function in (5.2),

$${}_8\varphi_7\left(\begin{matrix} \frac{acd}{q\alpha}, \sqrt{\frac{acd}{q^{1/2}\alpha}}, -\sqrt{\frac{acd}{q^{1/2}\alpha}}, q^{-\nu}, abcdq^{\nu-1}, \frac{a}{\alpha}, \frac{c}{\alpha}, \frac{d}{\alpha} \\ \sqrt{\frac{acd}{q\alpha}}, -\sqrt{\frac{acd}{q\alpha}}, \frac{q^\nu acd}{\alpha}, \frac{q^{1-\nu}}{\alpha b}, ac, ad, cd \end{matrix} ; q, q\frac{\alpha}{b} \right)$$
$$\underset{\nu\to\infty}{\longrightarrow}\ {}_6\varphi_5\left(\begin{matrix} \frac{acd}{q\alpha}, \sqrt{\frac{acd}{q^{1/2}\alpha}}, -\sqrt{\frac{acd}{q^{1/2}\alpha}}, \frac{a}{\alpha}, \frac{c}{\alpha}, \frac{d}{\alpha} \\ \sqrt{\frac{acd}{q\alpha}}, -\sqrt{\frac{acd}{q\alpha}}, ac, ad, cd \end{matrix} ; q, \alpha^2 \right)$$
$$= \frac{(\alpha a, \alpha c, \alpha d, acd/\alpha; q)_\infty}{(ac, ad, cd, \alpha^2; q)_\infty} \tag{5.3}$$

by (II.20) of [10]. Therefore, as $\nu \to \infty$,

$$\widetilde{v}_\nu\left(\frac{1}{2}(\alpha+\alpha^{-1}); a, b, c, d\right) = \frac{(\alpha a, q\alpha/b, \alpha c, \alpha d; q)_\infty}{(ac, ad, \alpha^2; q)_\infty} (q^{1-\nu}/\alpha b; q)_\infty [1 + \mathrm{o}(1)]. \tag{5.4}$$

But for the positive values of q and αb the function

$$(q^{1-\nu}/\alpha b; q)_\infty$$

oscillates and has an infinity of real zeros as ν approaches infinity (see [12] for details).

In a similar fashion, one can consider some other properties of zeros of the ${}_8\varphi_7$-function (5.1), (5.2) close to those established in [9, 12, 13] at the level of the basic trigonometric functions and the q-Bessel function, respectively. We will elaborate further on the properties of ν-solutions of (4.13) in our next paper [21].

Acknowledgments

I would like to thank Dick Askey for encouraging conversations and also for presenting my talk on a related topic at Montréal's meeting on 'Algebraic Methods and q-Special Functions' (May 1996). I wish to thank Joaquín Bustoz, George Gasper, Mourad Ismail, and Mizan Rahman for valuable discussions and comments. This paper was prepared while I visited the Department of Mathematics at Arizona State University and I gratefully acknowledge their hospitality.

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